31. Let $I = \int_0^{\pi/4} \log(1+\tan x)\,dx$ ………..(i)

$$I = \int_0^{\pi/4} \log\left(1+\tan\left(\frac{\pi}{4}-x\right)\right)dx$$

$\therefore$ $$I = \int_0^{\pi/4} \log\left(1+\frac{1-\tan x}{1+\tan x}\right)dx$$

$$I = \int_0^{\pi/4} \log\left(\frac{1+\tan x+1-\tan x}{1+\tan x}\right)dx$$

$$I = \int_0^{\pi/4} \log\left(\frac{2}{1+\tan x}\right)dx$$

$\Rightarrow$ $I = \int_0^{\pi/4} \log 2\,dx - I = 2I = \frac{\pi}{4}\log 2$ $\Rightarrow$ $I = \frac{\pi}{8}(\log 2)$

32. Let $I = \int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x}dx$

$$I = \int_{-\pi}^{\pi} \frac{2x}{1+\cos^2 x}dx + \int_{-\pi}^{\pi} \frac{2x\sin x}{1+\cos^2 x}dx$$

$\Rightarrow$ $I = I_1 + I_2$

Now, $I_1 = \int_{-\pi}^{\pi} \frac{2x}{1+\cos^2 x}dx$

$\therefore$ $$I_2 = \int_{-\pi}^{\pi} \frac{2x\sin x}{1+\cos^2 x}dx = 2.2\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$$

$$= 4\int_0^{\pi} \frac{(\pi-x)\sin(\pi-x)}{1+[\cos(\pi-x)]^2}\,dx$$

$$= 4\int_0^{\pi} \frac{(\pi-x)\sin x}{1+\cos^2 x}\,dx$$

$$= 4\int_0^{\pi} \frac{\pi\sin x}{1+\cos^2 x}dx - 4\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$$

$\Rightarrow$ $I_2 = 4\pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}dx - I_2$ $\Rightarrow$ $2I_2 = 4\pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}dx$

Put $\cos x = t \Rightarrow -\sin x dx = dt$

$\therefore$ $$I_2 = -2\pi\int_1^{-1} \frac{dt}{1+t^2}$$

$$= 2\pi\int_{-1}^{1} \frac{dt}{1+t^2} = 4\pi\int_0^{1} \frac{dt}{1+t^2} = \pi^2$$

$\therefore$ $I = I_1 + I_2 = 0 + \pi^2 = \pi^2$

33. Let $I=\int_{-1/\sqrt{3}}^{1/\sqrt{3}}\left(\frac{x^4}{1-x^4}\right)\cos^{-1}\left(\frac{2x}{1+x^2}\right)dx$ ....(i)

Put $x=-y \Rightarrow dx=-dy$

$\therefore \quad I=\int_{1/\sqrt{3}}^{-1/\sqrt{3}}\frac{y^4}{1-y^4}\cos^{-1}\left(\frac{-2y}{1+y^2}\right)(-1)dy$

Now, $\cos^{-1}(-x)=\pi-\cos^{-1}x$ for $-1\le x\le 1$.

$\therefore \quad I=\int_{-1/\sqrt{3}}^{1/\sqrt{3}}\frac{y^4}{1-y^4}\left[\pi-\cos^{-1}\left(\frac{2y}{1+y^2}\right)\right]dy \quad =\pi\int_{-1/\sqrt{3}}^{1/\sqrt{3}}\frac{y^4}{1-y^4}dy-\int_{-1/\sqrt{3}}^{1/\sqrt{3}}\frac{y^4}{1-y^4}\cos^{-1}\left(\frac{2y}{1+y^2}\right)dy$

$=\pi\int_{-1/\sqrt{3}}^{1/\sqrt{3}}\frac{x^4}{1-x^4}dx-\int_{-1/\sqrt{3}}^{1/\sqrt{3}}\frac{x^4}{1-x^4}\cos^{-1}\left(\frac{2x}{1+x^2}\right)dx$

$\Rightarrow \quad I=\pi\int_{-1/\sqrt{3}}^{1/\sqrt{3}}\frac{x^4}{1-x^4}dx-I$ [from equation (i)]

$\Rightarrow \quad 2I=\pi\int_{-1/\sqrt{3}}^{1/\sqrt{3}}\frac{x^4}{1-x^4}dx=\pi\int_{-1/\sqrt{3}}^{1/\sqrt{3}}\left[-1+\frac{1}{1-x^4}\right]dx$

$=-\pi\int_{-1/\sqrt{3}}^{1/\sqrt{3}}1dx+\pi\int_{-1/\sqrt{3}}^{1/\sqrt{3}}\frac{dx}{1-x^4} \quad =-\pi[x]_{-1/\sqrt{3}}^{1/\sqrt{3}}+\pi\cdot I_1$, where $I_1=\int_{-1/\sqrt{3}}^{1/\sqrt{3}}\frac{dx}{1-x^4}$

$\Rightarrow \quad 2I=-\pi\left(\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{3}}\right)+\pi I_1=-\frac{2\pi}{\sqrt{3}}+\pi I_1$

Now, $I_1=\int_{-1/\sqrt{3}}^{1/\sqrt{3}}\frac{dx}{1-x^4}=2\int_0^{1/\sqrt{3}}\frac{dx}{1-x^4}$ [since, the integral is an even function]

$=\int_0^{1/\sqrt{3}}\frac{1+1+x^2-x^2}{(1-x^2)(1+x^2)}dx \quad =\int_0^{1/\sqrt{3}}\frac{1}{1-x^2}dx+\int_0^{1/\sqrt{3}}\frac{1}{1+x^2}dx \;=\frac{1}{2}\ln(2+\sqrt{3})+\frac{\pi}{6}$

$\therefore \quad 2I=\frac{-2\pi}{\sqrt{3}}+\frac{\pi}{2}\ln(2+\sqrt{3})+\frac{\pi^2}{6} \quad =\frac{\pi}{6}[\pi+3\ln(2+\sqrt{3})-4\sqrt{3}] \;\Rightarrow \quad I=\frac{\pi}{12}[\pi+3\ln(2+\sqrt{3})-4\sqrt{3}]$

34. Let $I=\int_2^3\frac{2x^5+x^4-2x^3+2x^2+1}{(x^2+1)(x^4-1)}dx$

$I=\int_2^3\frac{2x^5-2x^3+x^4+1+2x^2}{(x^2+1)(x^2-1)(x^2+1)}dx \;=\int_2^3\frac{2x^3(x^2-1)+(x^2+1)^2}{(x^2+1)^2(x^2-1)}dx$

$=\int_2^3\frac{2x^3(x^2-1)}{(x^2+1)^2(x^2-1)}dx+\int_2^3\frac{(x^2+1)^2}{(x^2+1)^2(x^2-1)}dx \quad =\int_2^3\frac{2x^3}{(x^2+1)^2}dx+\int_2^3\frac{1}{(x^2-1)}dx \Rightarrow I=I_1+I_2$

Now, $I_1=\int_2^3\frac{2x^3}{(x^2+1)^2}dx$

Put $x^2+1=t \Rightarrow 2x\,dx=dt$

$\therefore \quad I_1=\int_5^{10}\frac{(t-1)}{t^2}dt=\int_5^{10}\frac{1}{t}dt-\int_5^{10}\frac{1}{t^2}dt \;=[\log t]_5^{10}+\left[\frac{1}{t}\right]_5^{10}$

$= \log 10 - \log 5 + \frac{1}{10} - \frac{1}{5} = \log 2 - \frac{1}{10}$

Again, $I_2 = \int_2^3 \frac{1}{(x^2-1)}\, dx = \frac{1}{2}\log\frac{2}{1} - \frac{1}{2}\log\frac{4}{3}$

From equation (i), $I = I_1 + I_2 = \log 2 - \frac{1}{10} + \frac{1}{2}\log 2 - \frac{1}{2}\log\frac{4}{3}$

$\frac{1}{2}\log 6 - \frac{1}{10}$

35. Since, $f(x)$ is a cubic polynomial. Therefore, $f'(x)$ is a quadratic polynomial and $f(x)$ has relative maximum and minimum at $x = \frac{1}{3}$ and $x = -1$ respectively, therefore, –1 and 1/3 are the roots of $f'(x) = 0$.

$\therefore \quad f'(x) = a(x+1)\left(x - \frac{1}{3}\right) = a\left(x^2 - \frac{1}{3}x + x - \frac{1}{3}\right) = a\left(x^2 + \frac{2}{3}x - \frac{1}{3}\right)$

Now, integrating w.r.t. x, we get

$f(x) = a\left(\frac{x^3}{3} + \frac{x^2}{3} - \frac{x}{3}\right) + c$ where, c is constant of integration.

Again, $f(-2) = 0$

$\therefore \quad f(-2) = a\left(-\frac{8}{3} + \frac{4}{3} + \frac{2}{3}\right) + c$

$\Rightarrow \quad 0 = a\left(\frac{-8+4+2}{3}\right) + c$

$\Rightarrow \quad 0 = \frac{-2a}{3} + c \Rightarrow c = \frac{2a}{3}$

$\therefore \quad f(x) = a\left(\frac{x^3}{3} + \frac{x^2}{3} - \frac{x}{3}\right) + \frac{2a}{3} = \frac{a}{3}(x^3 + x^2 - x + 2)$

Again, $\int_{-1}^{1} f(x)\,dx = \frac{14}{3}$ [given]

$\Rightarrow \quad \int_{-1}^{1} \frac{a}{3}(x^3 + x^2 - x + 2)\,dx = \frac{14}{3}$

$\Rightarrow \quad \int_{-1}^{1} \frac{a}{3}(0 + x^2 + 2)\,dx = \frac{14}{3}$ [$\because$ $y = x^3$ *and* $y = -x$ are odd functions]

$\Rightarrow \quad \frac{a}{3}\left[2\int_0^1 x^2\,dx + 4\int_0^1 1\,dx\right] = \frac{14}{3} \quad \Rightarrow \quad \frac{a}{3}\left[\left(\frac{2x^3}{3} + 4x\right)\right]_0^1 = \frac{14}{3}$

$\Rightarrow \quad \frac{a}{3}\left(\frac{2}{3} + 4\right) = \frac{14}{3} \Rightarrow \frac{a}{3}\left(\frac{14}{3}\right) = \frac{14}{3} \quad \Rightarrow \quad a = 3$

Hence, $f(x) = x^3 + x^2 - x + 2$

36. Let $I = \int_0^{\pi} \frac{x\sin(2x).\sin\left(\frac{\pi}{2}\cos x\right)}{(2x-\pi)} dx$ ....(i)

Then $I = \int_0^{\pi} \frac{(\pi - x).\sin 2(\pi - x).\sin\left[\frac{\pi}{2}\cos(\pi - x)\right]}{2(\pi - x) - \pi} dx$ ....(ii)

$\Rightarrow \quad I = \int_0^{\pi} \frac{(\pi - x).\sin 2x.\sin\left(\frac{\pi}{2}\cos x\right)}{\pi - 2x} dx$

$\Rightarrow \quad I = \int_0^{\pi} \frac{(x-\pi)\sin 2x.\sin\left(\frac{\pi}{2}\cos x\right)}{(2x-\pi)} dx$ ....(iii)

On adding equations (i) and (iii), we get $2I = \int_0^{\pi} \sin 2x.\sin\left(\frac{\pi}{2}\cos x\right) dx$

$\Rightarrow \quad I = \int_0^{\pi} \sin x\cos x.\sin\left(\frac{\pi}{2}\cos x\right) dx$ $\left[\text{put}\ \frac{\pi}{2}\cos x = t \Rightarrow -\frac{\pi}{2}\sin x dx = dt \Rightarrow \sin x dx = -\frac{2}{\pi} dt\right]$

$\therefore \quad I = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} \frac{2t}{\pi}.\sin t\, dt = \frac{4}{\pi^2}\int_{-\pi/2}^{\pi/2} t\sin t\, dt$

$\Rightarrow \quad I = \frac{4}{\pi^2}[-t\cos t + \sin t]_{-\pi/2}^{\pi/2} = \frac{4}{\pi^2} \times 2 = \frac{8}{\pi^2}$

37. We know that,

$2\sin x[\cos x + \cos 3x + \cos 5x + ... + \cos(2k-1)x]$

$= 2\sin x\cos x + 2\sin x\cos 3x + 2\sin x\cos 5x + ... + 2\sin x\cos(2k-1)x$

$= \sin 2x + (\sin 4x - \sin 2x) + (\sin 6x - \sin 4x) + ... + \{\sin 2kx - \sin(2k-2)x\} = \sin 2kx$

$\therefore \quad 2[\cos x + \cos 3x + \cos 5x + ... + \cos(2k-1)x] = \frac{\sin 2kx}{\sin x}$ ....(i)

Now, $\sin 2kx.\cot x = \frac{\sin 2kx}{\sin x}.\cos x$

$= 2\cos x[\cos x + \cos 3x + \cos 5x + ... + \cos(2k-1)x]$ [from equation (i)]

$= [2\cos^2 x + 2\cos x\cos 3x + 2\cos x\cos 5x + ... + 2\cos x\cos(2k-1)x]$

$= (1+\cos 2x) + (\cos 4x + \cos 2x) + (\cos 6x + \cos 4x) + ... + \{\cos 2kx + \cos(2k-2)x\}$

$= 1 + 2[\cos 2x + \cos 4x + \cos 6x + ... + \cos(2k-2)x] + \cos 2kx$

$\therefore \quad \int_0^{\pi/2} 1.dx + 2\int_0^{\pi/2} (\cos 2x + \cos 4x .... \cos(2k-2)x)dx + \int_0^{\pi/2} \cos(2k)x\, dx$

$= \frac{\pi}{2} + 2\left[\frac{\sin 2x}{2} + \frac{\sin 4x}{4} + ... + \frac{\sin(2k-2)x}{(2k-2)}\right]_0^{\pi/2} + \left[\frac{\sin(2k)x}{2k}\right]_0^{\pi/2} = \frac{\pi}{2}$

38. Let $I = \int_0^a f(a-x).g(a-x)dx = \int_0^a f(x).\{2-g(x)\}dx$ $[\because f(a-x) = f(x) \text{ and } g(x) + g(a-x) = 2]$

$= 2\int_0^a f(x)dx - \int_0^a f(x)g(x)dx$

$\Rightarrow \quad I = 2\int_0^a f(x)dx - I \quad \Rightarrow \quad 2I = 2\int_0^a f(x)\,dx \quad \therefore \quad \int_0^a f(x)g(x)dx = \int_0^a f(x)dx$

39. Let $I = \int_0^{2a} \frac{f(x)}{f(x) + f(2a-x)} dx$ ....(i)

$I = \int_0^{2a} \frac{f(2a-x)}{f(2a-x) + f(x)}$ ....(ii)

On adding equations (i) and (ii), we get

$$2I = \int_0^{2a} 1\, dx = 2a \Rightarrow I = a$$

40. Let $I = \int_0^1 \log(\sqrt{1-x} + \sqrt{1+x})dx$

Put $x = \cos 2\theta \Rightarrow dx = -2\sin 2\theta d\theta$

$\therefore \quad I = -2\int_{\pi/4}^0 \log[\sqrt{1-\cos 2\theta} + \sqrt{1+\cos 2\theta}](\sin 2\theta)d\theta$

$= -2\int_{\pi/4}^0 \log[\sqrt{2}(\sin\theta + \cos\theta)]\sin 2\theta \; d\theta$

$= -2\int_{\pi/4}^0 [(\log\sqrt{2})\sin 2\theta + \log(\sin\theta + \cos\theta).\sin 2\theta]d\theta =$

$= -2\log\sqrt{2}\left[\frac{-\cos 2\theta}{2}\right]_{\pi/4}^0 - 2\int_{\pi/4}^0 \log(\sin\theta + \cos\theta).\sin 2\theta.d\theta$

$= \log\sqrt{2} - 2\left[-\left\{\log(\sin\theta + \cos\theta).\frac{\cos 2\theta}{2}\right\}_{\pi/4}^0 - \int_{\pi/4}^0 \left(\frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta} \times \frac{-\cos 2\theta}{2}\right)d\theta\right]$

$= \log(\sqrt{2}) - 2\left[0 + \frac{1}{2}\int_{\pi/4}^0 (\cos\theta - \sin\theta)^2 d\theta\right]$

$= \frac{1}{2}\log 2 - \int_{\pi/4}^0 (1 - \sin 2\theta)d\theta = \frac{1}{2}\log 2 - \left[\theta + \frac{\cos 2\theta}{2}\right]_{\pi/4}^0 \quad = \frac{1}{2}\log 2 - \left(\frac{1}{2} - \frac{\pi}{4}\right) = \frac{1}{2}\log 2 - \frac{1}{2} + \frac{\pi}{4}$